ADVANCES IN MATHEMATICS 28, 231–232 (1978)

On the Nonomnipotence of Regular Summability Methods

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Let $T = (t_{m,n})$ $(m, n = 1, 2,...;$ all $t_{m,n} \geqslant 0)$ define a regular summability method. It is known [1] that there is a bounded divergent sequence whose T-transform is also divergent. Here we point out that one can say more: namely, that for some real, bounded, divergent sequence $\{a_n\}_{n=1}^{\infty}$, its T-transform diverges just a badly as itself. For every real sequence $\{a_n\}_{n=1}^{\infty}$, its T-transform $\{b_n\}_{n=1}^{\infty}$ satisfies [2],

$$\varliminf_{n\to\infty} a_n \leqslant \varliminf_{n\to\infty} b_n \leqslant \varlimsup_{n\to\infty} b_n \leqslant \varlimsup_{n\to\infty} a_n$$

so that, if $\{a_n\}$ is bounded,

$$\varlimsup_{n\to\infty} b_n - \varliminf_{n\to\infty} b_n \leqslant \varlimsup_{n\to\infty} a_n - \varliminf_{n\to\infty} a_n, \tag{*}$$

and, thus, the divergence of $\{b_n\}$ is not worse than that of $\{a_n\}$. Our goal is a real, bounded, divergent sequence $\{a_n\}$ for which equality holds in (*). As such a sequence one can take the sequence, consisting of 1's and -1's, defined in [1], as the argument given there does, in fact, establish the desired properties.

If the $t_{m,n}$ are not assumed $\geqslant 0$, but only real, matters are a bit worse. Inequality (*) is replaced by

$$\varlimsup_{n\to\infty} b_n - \varliminf_{n\to\infty} b_n \leqslant M(\varlimsup_{n\to\infty} a_n - \varliminf_{n\to\infty} a_n), \tag{**}$$

where

$$M = \varlimsup_{m\to\infty} \sum_{n=1}^{\infty} | t_{m,n} |$$

0001-8408/78/0283-0231$05.00/0

(M is necessarily finite and $\geqslant 1$). Equality in (**) can always be attained, by a divergent sequence $\{a_n\}_{n=1}^{\infty}$ of 1's and -1's. The construction is a modification of that in [1].

For general complex $t_{m,n}$, matters are different. With M as before, it is always possible to find a divergent sequence $\{a_n\}_{n=1}^{\infty}$ of numbers of absolute value 1 for which

$$\overline{\lim_{n,m\to\infty}} \mid b_n - b_m \mid = M \cdot \overline{\lim_{n,m\to\infty}} \mid a_n - a_m \mid,$$

which is the natural analog of attaining equality in (*) or (**). But sometimes even more can be done: Taking $\omega = -\frac{1}{2} + 3^{1/2}/2\, i$ ($\omega^3 = 1$), and any positive number K, we define a regular summability matrix $t_{m,n}$ by $t_{n,n} = 1$, $t_{m,n} = (-1)^m \cdot \omega^{-n} K$ for $n = m + 1, m + 2$, and $m + 3$; and $t_{m,n} = 0$ otherwise. Then $M = 3K + 1$. Setting $a_n = \omega^n$, we see that $b_n = \omega^n + (-1)^n \cdot 3K$. Thus

$$A = \overline{\lim_{n,m\to\infty}} \mid a_n - a_m \mid = 3^{1/2},$$

$$B = \overline{\lim_{n,m\to\infty}} \mid b_n - b_m \mid = [(6K + \tfrac{3}{2})^2 + \tfrac{3}{4}]^{1/2};$$

and $B/(AM)$ can be made arbitrarily close to $2/3^{1/2} > 1$ by choosing K sufficiently large.

References

1. B. R. Gelbaum and J. M. H. Olmsted, "Counterexamples in Analysis," pp. 66–68, Holden–Day, San Francisco, 1964.
2. G. H. Hardy, "Divergent Series," p. 52, Theorem 9, Oxford Univ. Press, Oxford, 1949.